Discrete Laplace Operator on Meshed Surfaces

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1 Introduction

The Laplace-Beltrami operator (manifold Laplacian) is a fundamental geometric object associated to a Riemannian manifold and has many desirable properties. In recent years a class of methods based on the discrete Laplace operator has been used for various tasks of geometric processing. Several discretizations of the Laplacian for an arbitrary mesh have been proposed in [5, 4]. However, a detailed theoretical analysis of existing discretizations in [6, 3] shows that while convergence can be established for special classes of meshes, none of these methods can be expected to converge for surface meshes in general, a finding, which is borne out by experimental results in Section 4 and [6].

In this paper we present the first algorithm to provide a discretization for the Laplace-Beltrami operator for an arbitrary meshed surface with strong convergence guarantees. The algorithm for computing the surface Laplacian is based on the functional approximation of the Laplace operator (see [2]) and is related to the set of algorithms for computing Laplacian of point clouds in data analysis and machine learning [1, 2]. An important difference is that in machine learning, the samples are usually believed to be drawn independently from a probability distribution and the convergence occurs with high probability. In surface modeling, however, vertices of a mesh typically are not sampled independently from a probability distribution but are generated by some deterministic process. Thus approximation guarantees need to be made for *all* sufficiently fine meshes, and probabilistic techniques based on the law of large numbers cannot be applied.

2 Notations

In order to obtain the theoretical results of this paper, we need a quantitative measure of how well a mesh approximates the underlying surface. A mesh K approximates the surface S if the vertices of K are in S. Let ρ be the reach of S. K is an (ϵ, η)-*approximation* of S, if the following conditions hold:

1. For each face $t \in K$, its diameter (i.e, the maximal distance between any two points on t) is at most $\epsilon\rho$.
2. For any face $t \in K$ and any vertex $p \in t$, the angle between vectors n_t and n_p, $\angle(n_t, n_p)$, is at most η.

We denote V the set of vertices of the mesh K. Mesh Laplacian takes a function $f : V \to \mathbb{R}$ as **input** and produces another function $\mathrm{L}_K^h f : V \to \mathbb{R}$ as **output**. L_K^h is called the **mesh Laplace operator**, and is computed, for any $w \in V$, as follows:

$$\mathrm{L}_K^h f(w) = \frac{1}{4\pi h^2} \sum_{t \in K} \frac{\mathrm{A}(t)}{\#t} \sum_{p \in V(t)} e^{-\frac{\|p-w\|^2}{4h}} (f(p) - f(w))$$

The theoretical results in this paper show that when K is a sufficiently fine mesh of a smooth underlying surface S, L_K^h is close to the surface Laplacian Δ_S. To connect the mesh Laplace operator L_K^h with the surface Laplacian Δ_S, we need an intermediate object called *functional Laplace operator* F_S^h. Given a point $w \in S$ and a function $f : S \to \mathbb{R}$, it is defined as:

$$\mathrm{F}_S^h f(w) = \frac{1}{4\pi h^2} \int_{x \in S} e^{-\frac{\|x-w\|^2}{4h}} (f(x) - f(w))\, d\nu(x).$$

3 Main Results

Approximating integral on a surface. We show that for any Lipschitz function on S, its integral over S can be approximated by its discretization $\mathrm{I_K} g$ defined as follows:

$$\mathrm{I_K} g = \sum_{t \in K} \frac{\mathrm{A}(t)}{\#t} \sum_{v \in V(t)} g(v). \quad (1)$$

Theorem 3.1 *Given an* (ε, η)*-approximation* K *of a smooth compact surface* S *with* $\varepsilon, \eta < 0.1$*, and any Lipschitz function* $g : S \to \mathbb{R}$*, let* ρ *be the reach of* S *and* $\mathrm{Lip}(g)$ *be the Lipschitz constant of function g,*

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i.e., $|g(x) - g(y)| < \mathrm{Lip}(g) d_S(x, y)$. *Let* $\|g\|_\infty = \sup_{x \in S} |g(x)|$. *We have that*

$$\left| \int_S g d\nu - \mathrm{I}_\mathrm{K} g \right| \leq 10\varepsilon(\rho \mathrm{Lip}(g) + \|g\|_\infty) A(S) \quad (2)$$

where $d_S(x, y)$ *denote the shortest geodesic distance between two points* $x, y \in S$.

Approximating Laplace operator. Our main result on approximating the Laplace operator is the following.

Theorem 3.2 *Let the mesh* K_ϵ *be an* (ϵ, η)*-approximation of* S*, with* $\eta < 0.1$. *Put* $h(\epsilon) = \epsilon^{\frac{1}{2.5+\alpha}}$ *for an arbitrary fixed positive number* $\alpha > 0$. *Then for any* $f \in C^3(S)$

$$\lim_{\epsilon \to 0} \sup_{K_\epsilon} \left\| \mathrm{L}_{K_\epsilon}^{h(\epsilon)} f - \Delta_S f \right\|_\infty = 0 \quad (3)$$

where the supremum is taken over all $(\epsilon, 0.1)$ *approximations of* S.

The proof of Theorem 3.2 relies on the following two theorems, connecting the mesh Laplacian operator to the functional Laplace operator and the functional Laplacian to the Laplace-Beltrami operator respectively.

Theorem 3.3 *Let* K *be an* (ϵ, η)*-approximation of* S*, with* $\varepsilon, \eta < 0.1$. *Given a function* $f \in C^1(S)$*, let* $\|f\|_\infty = \sup_{x \in S} |f(x)|$, $\|\nabla f\|_\infty = \sup_{x \in S} \|\nabla f(x)\|$ *and* ρ *denote the reach of surface* S. *We have that for any point* $w \in S$,

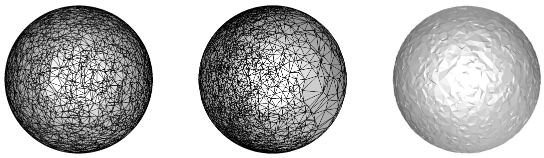
$$\begin{aligned} &|\mathrm{F}_S^h f(w) - \mathrm{L}_K^h(w)| \\ &\quad \leq \frac{5\varepsilon}{2\pi h^2} \left[2 \left(\frac{\rho}{\sqrt{h}} + 1 \right) \|f\|_\infty + \rho \|\nabla f\|_\infty \right] A(S). \end{aligned}$$

Theorem 3.4 ([2]) *For a function* $f \in C^3(S)$,

$$\lim_{h \to 0} \left\| \mathrm{F}_S^h f(w) - \Delta_S f(w) \right\|_\infty = 0 \quad (4)$$

4 Experiments

We compare our algorithm with the currently widely used COT scheme [4]. The experimental results show that COT scheme does not produce convergence for functions other than linear functions, and that our method is also robust with respect to noisy or non-uniformaly sampled data. Figure 1 shows some examples of three types of spherical meshes we experiment on. We define the error measure as $E = \frac{\|U - \widehat{U}\|}{\|U\|}$ where U and $\widehat{U}$ are evaluations of the surface Laplacian and the mesh Laplacian at vertices.



uniformly non-uniformly noisy

Figure 1: Meshes for spherical surface with 8000 sample points for meshes.

Method	500	2000	4000	8000	16000
uniformly					
COT	0.124	0.101	0.103	0.102	0.099
OUR	0.613	0.140	0.062	0.028	0.015
non-uniformly					
COT	0.100	0.079	0.074	0.075	0.072
OUR	0.447	0.156	0.076	0.038	0.019
noisy					
COT	0.308	1.271	1.448	2.631	0.817
OUR	0.612	0.153	0.069	0.043	0.018

Table 1: Results on sphere for $f = exp(x)$.

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