Size of Delaunay Triangulation for Points Distributed over Lower-dimensional Polyhedra: a Tight Bound

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Motivation. The Delaunay triangulation of a set of points is a data structure, which in low dimensions has applications in surface reconstruction (see Figure 1), mesh generation, molecular modeling, geographic information systems, and many other areas of science and engineering. Like many spatial partitioning techniques, however, it suffers from the "curse of dimensionality": the worst-case complexity of the Delaunay triangulation increases exponentially with the dimension. Indeed, the size of the Delaunay triangulation of n points in dimension d is at most $O(n^{\lceil \frac{d}{2} \rceil})$ and this bound is achieved exactly for points lying on the *moment curve*. The starting point of the work reported in this abstract is the following observation: all the examples that we have of point sets which have Delaunay triangulation of size $\Omega(n^{\lceil \frac{d}{2} \rceil})$ are distributed on one-dimensional curves. At the opposite extreme, points distributed uniformly inside a d-dimensional ball have Delaunay triangulations of complexity $O(n)$. Our motivation for this work is to fill in the picture for distributions between the two extremes, in which the points lie on manifolds of dimension $2 \leq p \leq d-1$.

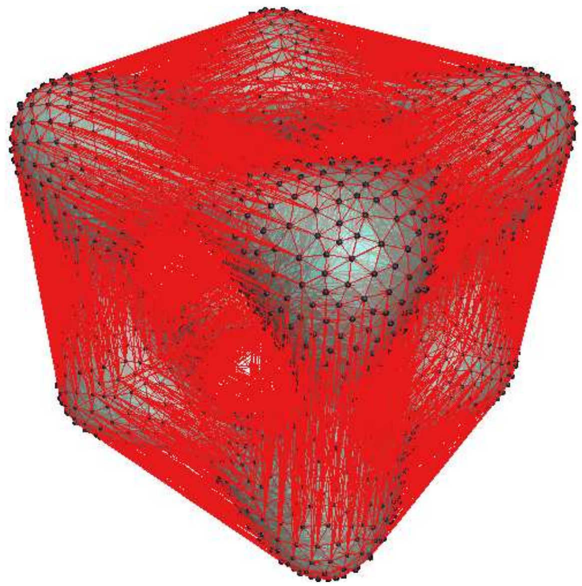

Figure 1: For clarity, we are only displaying the edges of the Delaunay triangulation of a set of points in $\mathbb{R}^3$ and the surface reconstructed from the Delaunay triangulation.

Main result. As an easy first case, we consider a p-*dimensional polyhedron* $\mathbb{P}$, that we define as the underlying space of a geometric simplicial complex of dimension p. Our point set S is a *sparse* ε*-sample* from $\mathbb{P}$. Sparse ε-sampling is a model, in which the sampling can be neither too sparse nor too dense. More precisely, we assume the following two conditions on S: (1) every point lying on a face F of $\mathbb{P}$ is at distance at most ε to a point in $S \cap F$; (2) every ball of radius ε contains a constant number of points in S. Let n be the number of points in S. We consider how the complexity of the Delaunay triangulation of S grows, as $n \to \infty$, while $\mathbb{P}$ remains fixed. Our main result is that the number of simplices of all dimensions is $O(n^{\frac{d-k+1}{p}})$ with $k = \lceil \frac{d+1}{p+1} \rceil$. This bound is tight and improves on the worst-case bound of $O(n^{\lceil d/2 \rceil})$ for all $2 \leq p \leq d-1$. It coincides with it for $p = 1$. The hidden constant factor depends, among other things, on the geometry of $\mathbb{P}$, which is constant since $\mathbb{P}$ is fixed. While our result is purely combinatorial, it has immediate algorithmic implications since there exist output sensitive algorithms to compute the Delaunay triangulation [8, 4].

Prior work. The complexity of the Delaunay triangulation of a set of points on a two-manifold in $\mathbb{R}^3$ has received considerable recent attention, since such point sets arise in practice, and their Delaunay triangulations are found nearly always to have linear size. Golin and Na [7] proved that the Delaunay triangulation of a large enough set of points distributed uniformly at random on the surface of a fixed con-

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vex polytope in $\mathbb{R}^3$ has expected size $O(n)$. They later [6] established an $O(n \lg^6 n)$ upper bound with high probability for the case in which the points are distributed uniformly at random on the surface of a non-convex polyhedron.

Attali and Boissonnat considered the problem using a sparse ε-sampling model similar to the one we use here, rather than a random distribution. For such a set of points distributed on a polygonal surface $\mathbb{P}$, they showed that the size of the Delaunay triangulation is $O(n)$ [2]. Our proof gives the same bound, and is perhaps a little simpler; but, our definition of sparse ε-sampling for polyhedra is a little more restrictive. In a subsequent paper with Lieutier [3] they considered "generic" surfaces, and got an upper bound of $O(n \lg n)$. Specifically, a "generic" surface is one for which each medial ball touches the surface in at most a constant number of points.

The genericity assumption is important. Erickson considered more general point distributions, which he characterized by the *spread*: the ratio of the largest inter-point distance to the smallest. The spread of a sparse ε-sample of n points from a two-dimensional manifold is $O(\sqrt{n})$. Erickson proved that the Delaunay triangulation of a set of points in $\mathbb{R}^3$ with spread Δ is $O(\Delta^3)$. Perhaps even more interestingly, he showed that this bound is tight for $\Delta = \sqrt{n}$, by giving an example of a sparse ε-sample of points from a cylinder that has a Delaunay triangulation of size $\Omega(n^{3/2})$ [5]. Note that this surface is not generic and has a degenerate medial axis.

The result presented in this abstract improves a bound established by the authors in [1] for $p > 3$.

Overview of the proof. We first define a subset of the medial axis of $\mathbb{P}$ called the *essential ε-quasi k-medial axis*. We show that this subset has dimension at most $d-k+1$ and that its $(d-k+1)$-dimensional volume is bounded from above by a constant that does not depend on ε. It follows that we can construct an ε-sample M of the essential ε-quasi k-medial axis with $m = O(\varepsilon^{-(d-k+1)})$ points. We associate to each point $z \in M$ a cover defined as

$$\mathrm{Cover}(z) = \bigcup_{x \in \Pi(z)} B(x, 6d\varepsilon),$$

where $\Pi(z)$ is the set of orthogonal projections of z onto the flats supporting faces of $\mathbb{P}$. For $k = \lceil \frac{d+1}{p+1} \rceil$, we then map each Delaunay simplex σ to a sample point z in M in such a way that the vertices of σ are contained in the cover of z. Since the cover of each point $z \in M$ contains a constant number of points in S, each point $z \in M$ is charged for a constant number of Delaunay simplices. It follows that the size of the Delaunay triangulation is bounded from above by the size of M which is $m = O(\varepsilon^{-(d-k+1)})$. Since S is a sparse ε-sample from a p-dimensional polyhedron, its cardinality is $n = \Omega(\varepsilon^{-p})$. Eliminating ε gives $m = O(n^{\frac{d-k+1}{p}})$.

The bound is tight. Let $k = \lceil \frac{d+1}{p+1} \rceil$. We construct a p-dimensional polyhedron in $\mathbb{R}^d$ out of a d-dimensional simplex by considering a collection of k pairwise disjoint faces $\{F_1, \ldots, F_k\}$ such that the first $k-1$ faces have dimension p. Writing q for the dimension of the last face, we have $(k-1)p + q = d - k + 1$. Consider now a sparse ε-sample S of $\mathbb{P} = \bar{F}_1 \cup \ldots \cup \bar{F}_k$. The $(k-1)$ first faces receive $\Omega(\varepsilon^{-p})$ sample points and the last face receives $\Omega(\varepsilon^{-q})$ sample points. Consider a $(k-1)$-dimensional simplex obtained by picking a sample point x_i on each face F_i. There exists a unique $(d-1)$-sphere tangent to F_i at x_i for $1 \leq i \leq k$. It follows that $\{x_1, \ldots, x_k\}$ is a Delaunay simplex and the amount of $(k-1)$-dimensional Delaunay simplices that we can construct this way is at least

$$\Omega(\varepsilon^{-p} \times \ldots \times \varepsilon^{-p} \times \varepsilon^{-q}) = \Omega(\varepsilon^{-(d-k+1)}).$$

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