A tight bound for the Delaunay triangulation of points on a polyhedron

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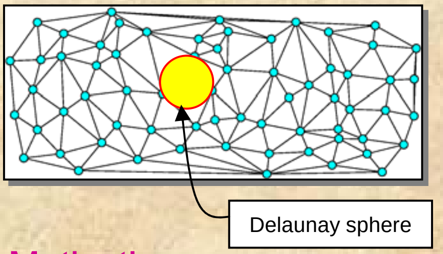
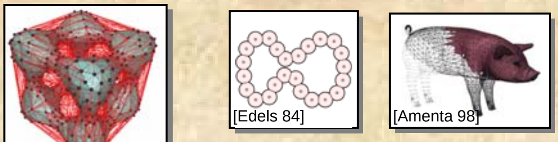
Delaunay triangulations

Let S be a set of n points in $\mathbf{R}^d$.
The Delaunay triangulation of S, Del(S), consists of simplices $\sigma \subseteq S$ such that there exists a sphere through σ and enclosing no points of S in its interior.

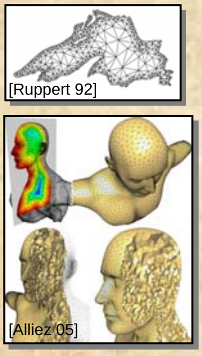


Applications in low dimension

Shape reconstruction



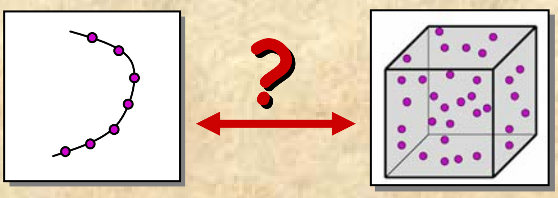
Mesh generation



Medial axis approximation, shape analysis

Motivation

The complexity of Del(S) is $O(n^{\lceil d/2 \rceil})$ [McMullen70].



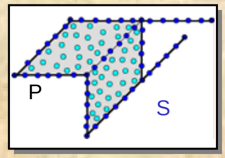
Moment curve: $\Theta(n^{\lceil d/2 \rceil})$ d-Volume: O(n)

Prior results for points distributed on surfaces in $\mathbf{R}^3$:

- Cylinder: $\Theta(n\sqrt{n})$ [Erickson 01]
- Smooth generic surface: O(n log n) [Att Boi Lie 03]
- Polyhedron: O(n) [Attali Boissonnat 02]

Main result

S is a λ-sparse ε-sample of a polyhedron P if:
(1) Every point in a face F of P is at distance at most ε to a point in $S \cap F$.
(2) Every d-ball with radius ε contains at most λ points of S.
S has $n = \Omega(\varepsilon^{-p})$ points.



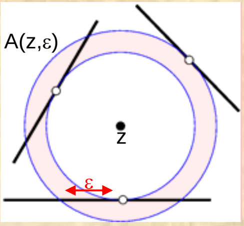
Let S be a λ-sparse ε-sample of a p-dimensional polyhedron in $\mathbf{R}^d$.

$$\text{Size}(\text{Del}(S)) = O\left(n^{\frac{d-k+1}{p}}\right) \quad \text{with } k = \left\lceil \frac{d+1}{p+1} \right\rceil$$

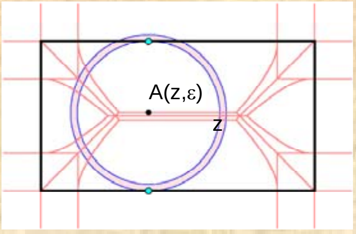
Outline of the proof

Essential ε-quasi k-medial axis:

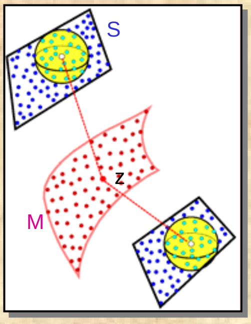
Variant of the medial obtained:
(1) by keeping strata with dimension at most d-k+1;
(2) by lopping branches that go to infinity.
We prove that its i-dimensional strata have a i-dimensional volume bounded by a constant that does not depend on ε.
We construct an ε-sample M of the essential ε-quasi k-medial with $m = O(\varepsilon^{-(d-k+1)})$ points.



largest P-empty ε-thin annulus centered at z.



The ε-quasi k-medial is the set of z / $A(z,\varepsilon)$ is tangent to at least k faces.



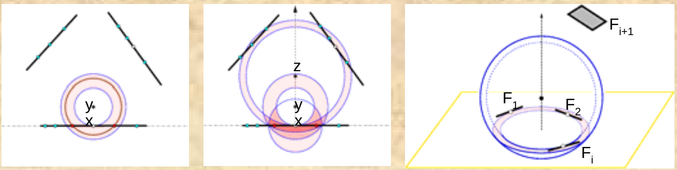
We map each $\sigma \in \text{Del}(S)$ to a sample z in M with

$$\text{Vertices}(\sigma) \subseteq \bigcup_{z \in \Pi(z,P)} B(z, 5d\varepsilon)$$

Each sample z in M is charged for a constant number of $\sigma \in \text{Del}(S)$.

$\Rightarrow$ size(Del(S)) = O(m).

Mapping a simplex to a point on ε-quasi k-medial axis:



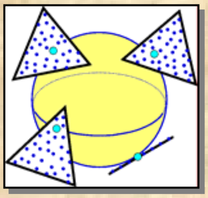
Sampling condition:
Delaunay sphere centered at y contained in $A(y,\varepsilon)$.

Key observation:
(k-1) faces of P are always contained in a hyperplane.

The bound is tight

P: k affinely independent simplices that span $\mathbf{R}^d$ with dimensions $q_1, \ldots, q_k$ and max q_i = p.

Size(Del(S)) = $\Omega(\varepsilon^{-q_1} \times \varepsilon^{-q_2} \times \ldots \times \varepsilon^{-q_k})$



Questions

- Can we relax the sampling condition?
- What if the sample points are noisy?
- Results for a random sampling model?
- What if the surface has curvature ?